NUMBER THEORY.

249. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A perfect number is a number which is equal to the sum of all its different divisors. In an old book on mathematics, the following method is given without proof for determining perfect numbers. The number $2^{n-1}(2^n - 1)$ is a perfect number if $2^n - 1$ is a prime number. Prove the formula.

250. Proposed by JOSEPH E. ROWE, State College, Pa.

Show by comparatively elementary means that the equation $x^{2n} + y^{2n} = z^{2n}$ is impossible of solution in positive integers x, y, z, and n, unless at least one of the integers $x, y, z \equiv 0 \pmod{3.}$. In particular, consider the case $n = 1$.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

450. Proposed by J. E. ROWE, Pennsylvania State College.

If the four roots of the quartic equation, $A \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, are so related that $B \equiv a_0a_4 - 4a_1a_3 + 3a_2^2 = 0$, show by elementary algebra that two roots of A are real and two imaginary. Show also by means of elementary algebra that A cannot have two equal roots without having three, if the condition $B = 0$ is satisfied.

Solution by J. A. Bullard, Worcester, Mass.

Let

$$A \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 \equiv (ax^2 + 2bx + c)(a'x^2 + 2b'x + c') = 0.$$

Then

$$a_0 = a'a, \quad 2a_1 = a'b + ab', \quad 6a_2 = a'c + 4b'b + ac', \quad 2a_3 = b'c + bc' \quad \text{and} \quad a_4 = c'c;$$

whence,

$$\begin{aligned} a_0a_4 &= a'ac'c, \\ -4a_1a_3 &= -ab'^2c - a'b^2c' - a'b'bc - abb'c', \\ 3a_2^2 &= \tfrac{1}{12}(a'c)^2 + \tfrac{4}{3}(b'b)^2 + \tfrac{1}{12}(ac')^2 + \tfrac{2}{3}a'b'bc + \tfrac{2}{3}abb'c' + \tfrac{1}{6}aa'cc'. \end{aligned}$$

Adding these equations and simplifying, we have

$$\text{(I)} \qquad B \equiv (b^2 - ac)(b'^2 - a'c') + \tfrac{1}{3}[bb' - \tfrac{1}{2}(a'c + ac')]^2.$$

If the original coefficients are real numbers then a, b, c, a', b', c' can always be taken so as to have real values. If $B = 0$ and the roots are all distinct the product of the discriminants $b^2 - ac$ and $b'^2 - a'c'$ is negative and hence one discriminant is negative and the other positive. Thus two of the four roots of $A = 0$ are the real and unequal roots of one quadratic and the other two roots are the imaginary roots of the second quadratic.

If $B = 0$ and two roots are equal let us assume them to be the roots of $ax^2 + 2bx + c = 0$, that is, $b^2 - ac = 0$. Then from (I) it follows that $bb' - \frac{1}{2}(a'c + ac') = 0$. Solving the quadratics and substituting from the relations just stated, we find the roots to be $-\dfrac{b}{a}$, $-\dfrac{b}{a}$, $-\dfrac{b}{a}$, $-\dfrac{bc'}{a'c}$. Thus, if $B = 0$ and two roots are equal, three roots must be equal.

A further examination of (I) shows that

If $B < 0$, two roots are imaginary and two are real and unequal.

If $B = 0$, two roots are imaginary and two are real and unequal, or three are equal, or all are equal.

If $B > 0$, two are imaginary and two are equal, or all are imaginary, or all are real. In this case we may have two double roots.

Also solved by George W. Hartwell and the Proposer.

451. Proposed by H. S. UHLER, Yale University.

Prove that

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cos\frac{x}{2^4}\cdots.$$